Contemporary Advances in Physics, XIX.

Fusion of Wave and Corpuscle Theories.

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In this article certain of the simple and familiar phenomena of optics and of electronics—for instance, refraction at a boundary between two media, and diffraction by a grating—are interpreted by *both* of the theories, undulatory and corpuscular, which have so often been condemned as incompatible with one another; the attitude being, that the theories may be brought into concordance by modifying one at least in ways which, extraordinary as they seem, do not quite destroy its character.

NOT quite five years ago I published in this journal an article entitled *Waves and Quanta*, expounding there the data which invited a corpuscular theory of light, regardless of the great array of classical phenomena of optics which demanded with no less insistence the long-triumphant undulatory theory. Today, not only are those data still extant and undeniable; they have been reinforced by observations on electron-streams which have compelled a wave-theory of free negative electricity, despite the very abundant evidence for free corpuscular electrons. Most physicists expect that not only light and negative electricity, but whatever other fundamentals there may be—meaning, probably, positive electricity and nothing else—will be found to conform in some ways to simple wave-theory, and in some to simple particle-theory. Most physicists, I think, would concede that the two ideas must be forced into one scheme, whatever violence it may entail to others of our preconceptions, inborn or inbred. We must stretch the theories and our minds, so that corpuscles and waves shall appear no longer as alternatives of which election must be made, but as complementary aspects of one reality.

To make a beginning with this process of stretching. I propose to treat some of the very simplest and most familiar of the phenomena, which up to lately have been interpreted by *one only* of the theories: phenomena such as the refraction of light in passing from air to water, the bending of the paths of electrons in passing from vacuum into metal, the diffraction of light and electrons from a ruled diffraction-grating. (None of these examples, incidentally, involves a theory of the structure of the atom.) Each of them shall be interpreted by the *other* theory—not in order to substitute the *other* for the *one*, but in order to practice the art of using *both* theories in alliance.

Refraction of Waves and Refraction of Corpuscles.

I presume that every textbook of optics and every history of physics informs its readers that anciently there was a controversy between a wave-theory of light (attributed to Huyghens) and a corpuscular theory (accredited to Newton) which was totally decided in 1850 by an experiment of Foucault. Light is refracted toward the normal in passing from air to water, and should therefore move more rapidly in water than in air if it consists of particles, but not so rapidly if it consists of waves—so runs the argument. Foucault and Fizeau discovered that light does move less rapidly in water than in air.[1] Let us analyze the argument more closely before deciding what was proved.

The reasoning from the "wave-theory" is usually made in graphic fashion by showing "Huyghens' construction" (Fig. 1) which should remind many a reader of his high school days! This is a very crude form of wave-theory, much too primitive to account for most of the phenomena which the physicist has in mind when he says that light (or electricity, or matter) is of the nature of waves; but for the present purpose it will do.

In Fig. 1, AA' is the trace, on the plane of the paper, of a wavefront moving through air (say) in the direction LM toward the boundary between air and water. It is the trace of the wavefront at a particular moment, say t; at a later moment, say t', the front has moved on to another position BB'. Denote by v the speed of the wavefront in air; then the perpendicular distance between BB' and AA' is equal to $v(t' - t)$. While the wave is advancing through this distance, its intersection with the boundary of the water sweeps over the distance AB, which we will denote by D. Designate by θ the angle between wavefront and boundary, the "angle of incidence." From the diagram one sees immediately:

$$\sin \theta = v(t' - t)/D. \tag{1}$$

Now in Huyghens' view, whenever the oncoming wavefront passed over an atom in the boundary-surface it incited that atom to emit a "wavelet." The circles drawn around various points on the line AB are the traces on the plane of the paper, of halves of those spherical wavelets—the halves expanding downwards into the water. Accord-

[1] Foucault usually gets all the credit, but Fizeau and Bréguet were working at the same time, incited by the same suggestion of Arago, and using the same method with differences in detail; and they announced their result only six weeks later. Indeed, at the meeting of the Académie des Sciences (May 6, 1850) at which Foucault reported his success, Fizeau said that if the sun had shone that day or the day before, they too would have had data to present.

ing to "Huyghens' Principle" the ongoing wavefront in the water is the envelope of these spheres. In Fig. 1 they and the ongoing wavefront are represented for the moment t' when the wave in the air reaches B. The radius AC of the wavelet expanding from A is then the distance which light traverses in water during time $(t' - t)$, for that wavelet started when the wave in the air reached A. Denote by v' the speed of light in water and by θ' the angle between the new

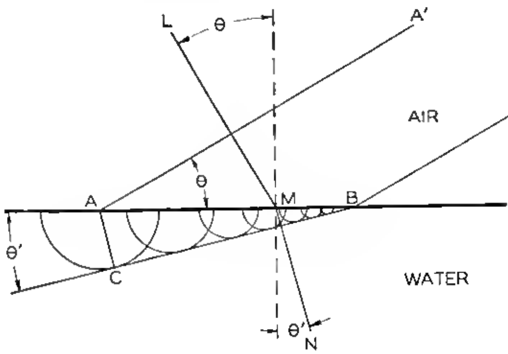

Fig. 1.

wavefront and the boundary, the "angle of refraction"; then from the diagram:

$$\sin \theta' = v'(t' - t)/D \tag{2}$$

and from (1) and (2) together, we obtain:

$$\sin \theta/\sin \theta' = v/v'. \tag{3}$$

From this familiar equation it follows in general, that the ratio $(\sin \theta/\sin \theta')$ is independent of the angle of incidence. (It is called the *index of refraction* of the second medium with respect to the first; I denote it hereafter by N.) Also it follows in particular, that when light is refracted towards the normal the wavefronts must move more slowly in the second medium than in the first, which is what Foucault verified, or rather, thought he had verified.

Now try it by the corpuscle-theory. In Fig. 1, I have the line LMN redrawn as a heavy line, and the lines at right angles to it left out; for the line LMN, one of the "rays" of light, is now to be interpreted as the path of a corpuscle, and there are no wavefronts.

So long as the corpuscle is too far from the boundary-surface to feel any force from the water, it moves in a straight line with unchanging momentum; for the forces exerted on it by the air, being equally applied in all directions, balance one another out. In the region near

the boundary, this remains the truth for the components of force parallel to the surface; but the components along the normal, applied respectively from the direction towards the air and the direction towards the water, need not be perfectly equal. After the corpuscle has gone through the transition region and reached the depths of the water, it continues in a straight line with a momentum of which the component parallel to the boundary—the "tangential" component, say—is still the same as it was in the air, while the normal component is changed. Denote by p_t and p_n these two components of the original momentum of the particle through the air, by p the magnitude of their resultant which is the original momentum; by p_t', p_n' and p' the corre-

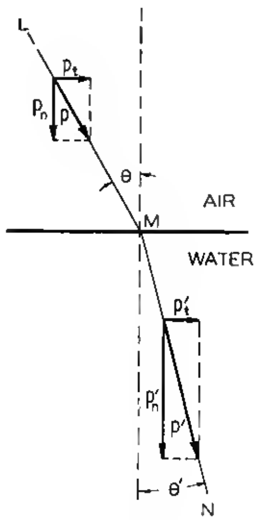

Fig. 2.

sponding quantities for the final flight of the corpuscle through the water. From Fig. 2 we see:

$$\sin \theta = p_t/\sqrt{p_t^2 + p_n^2} = p_t/p, \quad \sin \theta' = p_t'/p, \tag{4}$$

and since $p_t = p_t'$:

$$\sin \theta/\sin \theta' = p'/p. \tag{5}$$

The corpuscle-theory therefore leads to the statement that the sines of the angles of incidence and refraction stand to one another as the momenta of the corpuscle in the first medium and in the second; and when light is refracted towards the normal, the corpuscles must move with a greater momentum in the second medium than in the first.

Comparing the equations (5) and (3) to which the two conceptions lead, one sees that far from contradicting one another, they are both acceptable, provided that:

$$p/p' = v'/v. \tag{6}$$

We may hold both the theories simultaneously, we may interchange the two at will, provided we assume that the momentum of the corpuscles varies inversely as the speed of the wavefronts. In spite of the outcome of Foucault's experiment, we may adopt either the wave-theory or the corpuscle-theory or both at once to describe refraction, provided we assume that when a beam of light is refracted toward the normal, the speed of the wavefronts diminishes but the momentum of the corpuscles grows greater.

Why then did everyone concede that the corpuscular theory of light was killed by the experiment of Foucault? Because everyone was making two assumptions which seemed so obvious as to be hardly worth the stating, and so certain that it would have been regarded as absurd to call either into question:

(*A*) It was being assumed, that the momentum of a corpuscle must always be strictly proportional to its velocity; in other words, that the mass of a corpuscle must be invariant.

(*B*) It was being taken for granted that in a wave-theory of light the speed of the waves, and in a corpuscle-theory of light the speed of the corpuscles, must be identified with the actual speed of light as measured in any actual experiment.

When these assumptions are made, equation (5) goes over into the form,

$$\sin \theta / \sin \theta' = p'/p = v'/v, \tag{7}$$

which is contradictory to equation (3) and disproved by the experiment of Foucault.

But it no longer seems radical to change the first of these assumptions, for it is known from observation that there are particles—electrons, for example—of which the mass is not invariant, but depends upon the speed. For such a particle the momentum is not exactly proportional to the velocity. It is then not quite so revolutionary to go further, and suppose that the corpuscle of light is of so strange a nature that its velocity and its momentum are in magnitude inversely proportional to one another. If one made this supposition then one could accept the second assumption, and still explain the refraction of light by the corpuscle-theory.

Even the second assumption, however, is not sacred. It may seem absurd to set up a wave-theory of light, and then say that the speed of

the wavefronts is not to be identified with the measured speed of light. It does seem absurd to set up a corpuscle-theory, and then say that the speed of the corpuscles is not necessarily the same as that of light. Yet it may turn out in the end that a theory of either kind is strengthened, and made more competent to account for a variety of facts, by abandoning that easy and natural identification. I will try to prove by actual examples that it does so turn out. Meanwhile I summarize this section in a sentence:

If we wish to interpret light, or electricity, or matter, by both a corpuscle-theory and a wave-theory, the momentum of the corpuscles must be supposed to vary inversely as the speed of the waves.

I have omitted the special reference to refraction, for any more general theory must include that particular case, or fall down completely; I have added allusions to electricity and matter, for the test of any alteration of the two classical assumptions will depend chiefly on whether it helps in understanding the wavelike properties of these two, and not of light alone.

We now carry the wave-theory a great step beyond the primitive form in which Huyghens left it, by introducing the ideas of *frequency and wave-length.*

Wave-length of Waves and Momentum of Corpuscles

Instead of the single "wavefront" of Fig. 1, suppose a train of sine-waves of frequency ν, period $T(= 1/\nu)$, wave-length λ and wave number $\mu(= 1/\lambda)$ travelling through air along the course LMN. For definiteness, think of sound-waves. The condensation[1] of the air conforms to the equation:

$$\rho = \rho_0 \sin 2\pi\ (\nu t - \mu s + \alpha), \tag{8}$$

wherein s stands for distance measured from some arbitrary plane perpendicular to LM, and α for some constant. I write the equation down because one like it (or more than one) occurs in every wave-theory. In that of light there are six such equations, with components of electric and magnetic field strength replacing ρ; but it will be sufficient to think of one. In the wave-theory of matter there is one, with a quantity of very abstract meaning replacing ρ.

Now when the wave train passes through into the water, its frequency remains the same. With sound-waves, or any mechanical vibrations of matter, this is obvious; two pieces of matter in continuous contact must vibrate in unison, or not at all. We generalize this statement to cover light-waves, and waves of other varieties later

[1] The excess of the density over the normal value, divided by the normal value.

to be considered. Using primes to designate the values which things have in the second medium, we put:

$$\nu' = \nu. \tag{9}$$

The speed of the waves is the product of their wave-length by their frequency:

$$v = \nu\lambda, \qquad v' = \nu'\lambda'; \tag{10}$$

consequently:

$$v'/v = \lambda'/\lambda. \tag{11}$$

The wave-lengths of the wave train on the two sides of the boundary vary directly as the speeds.

Return now to the last section, and introduce this result into equation (6); one gets:

$$p'/p = \lambda/\lambda' \tag{12}$$

which means: we can interpret refraction of light (or of electricity, or of matter) by both the wave-theory and the corpuscle-theory, provided that we make the momentum of the corpuscle vary inversely as the wave-length of the waves.

Write accordingly,

$$p\lambda = \text{constant}. \tag{13}$$

Now there are several remarkable experiments which show that this relation actually holds, and moreover that the constant which appears in it is the universal constant h of Planck:

$$p = h/\lambda. \tag{14}$$

For instance, one may pour a stream of X-rays—that is to say, high-frequency light—into a gas, after having measured its wave-length in the known and reliable way depending on one of the phenomena in which X-rays behave as waves. A certain portion of the rays is scattered; it is scattered as though it consisted of corpuscles, each of which strikes an individual free electron and bounces off, the electron meanwhile recoiling from the blow.[2] Further analysis of the data shows that there is conservation of momentum—that the momentum which the electron gains is equal to that which the corpuscle of light has lost, *provided that the momentum of this latter is equal to the quotient of h by the wave-length of the rays.* For the wave-length of the scattered X-rays, measured in the same way as that of the primary rays was measured, is not the same as theirs; and the difference between the values of h/λ, before and after scattering, is equal to the momentum which the electron received.

[2] The Compton effect (cf. the seventh article of this series).

Again, one may pour a stream of electrons against a crystal or an optical ruled grating, after having measured the speed of the electrons in one of the well-known ways depending ultimately on the deflection of such a beam in known electric and magnetic fields.[3] The mass of the electrons being known, one knows also their momentum. Now the crystal or the grating, whichever it may be, forms from the primary beam a diffraction-pattern of new beams. Well! the formation of a diffraction-pattern is the primary reason for saying that light is wavelike, and it gives the primary way of measuring wave-length of light. One is equally obliged to admit that a stream of free negative electricity is wavelike, and to accept the value for its wave-length which the diffraction-pattern gives. Again it turns out that the wave-length is equal to the quotient of h by the momentum of the electrons.

It may be objected that in all of those experiments, the corpuscles were observed in a vacuum. Compton measured X-rays before and after scattering, but during the measurements they were in vacuum or at any rate in air. Davisson and Germer, Thomson and Rupp, observed electrons returning through the same evacuated space as they had crossed on their way to the diffracting lattice. One might emphasize that all these savants compared momenta and wave-lengths for different beams in the same medium instead of comparing them for the same beam in different media. The distinction is certainly worth noticing; but happily there are experiments which bear directly on refraction. Davisson and Germer measured, not precisely the refraction of an electron-stream passing from vacuum into nickel, but a minor perturbation of the diffraction-pattern which is due to that refraction. We will analyze their result, for nothing shows more clearly the relations—or lack of relation, the reader may think—between speed of waves, speed of corpuscles and measured speed of stream.

Davisson and Germer came to values of the index of refraction ($\sin\theta/\sin\theta'$) which were greater than unity—which corresponded therefore to a bending of the stream towards the normal, as it passed from vacuum into nickel—which therefore signified that the speed of the waves is not so great in nickel as in air.

On the other hand, it is known that when an individual electron passes from vacuum into a metal, its kinetic energy and its velocity increase as it goes through the surface. We have in fact the situation described in the corpuscle-theory picture of refraction, a few pages back. Return to equations (4) and (5), and consider a corpuscle for

[3] The experiments of Davisson and Germer, of G. P. Thomson, and of Rupp (cf. the eighteenth article of this series).

which the momentum p, the velocity u, the kinetic energy K, the mass m are related to one another as in Newtonian mechanics—properties which are practically those of electrons except when these are moving much more rapidly than any involved in these experiments:

$$p = mu, \qquad K = \tfrac{1}{2}mu^2. \tag{15}$$

Use u_t and u_n to denote tangential and normal components of speed; use primes to designate the values which things have in the second medium (nickel). Starting from equation (5), we continue:

$$\sin\theta/\sin\theta' = N = M'/M = u'/u;$$
$$N^2 - 1 = (u'^2 - u^2)/u^2 = (K' - K)/K. \tag{16}$$

The quantity $(K' - K)$ is the gain in kinetic energy which the electron wins on passing into the nickel; and this gain, as I have said, is positive; hence by equation (16) the index of refraction must be greater than unity. This is in agreement with the result of Davisson and Germer; the agreement, in fact, appears to be quantitative.[4]

It is always pleasant to get an agreement; but note how we got this one. We got it by dropping the assumption that the speed of the corpuscles and the speed of the waves must be the same. Or rather, by not making that assumption. For though the fact of experience is always the same—the swerving of the electron-stream *toward* the normal as it enters the nickel—it is interpreted by the two theories in opposite ways; the waves are slowed down, but the corpuscles are speeded up, in passing from the vacuum to the metal. Even if wave-speed and corpuscle-speed were the same in empty space, they could not be the same in any other medium.

This is more serious than it may appear at first. It amounts in effect to saying that a beam of free negative electricity has two different speeds; one when we visualize it as a jet of particles, another quite different when we visualize it as a train of waves.

But is not one of these "the right one" and the other "a wrong one," and can we not settle between them by measuring the actual time which the electricity takes to pass a measured distance? Let us examine this possibility. We shall find that after all it is not so easy to evade the ambiguity in such a fashion.

Phase-Speed and Group-Speed

Suppose an endless train of perfect monochromatic sine-waves marching along through space. For definiteness, think again of sound-

[4] There is a remarkably interesting correlation between these results and the new statistical theory of the electron-gas inside the metal (cf. my article in the October 1929 number of this *Journal*, pp. 710–716).

waves. It might seem as if we could measure their speed by picking out one crest, as A of Fig. 3, and checking off with a stop-watch the moments when it passes two fixed markers placed a known distance apart. Not so; for we cannot see or hear or in any way perceive the individual crests. The wave train produces a perfectly uniform tone in the ear which it strikes. If two listeners are stationed at different points along the path of the sound, neither can recognize the moment at which any particular crest glided by. All they can recognize, all they can compare, is the moment of passage of a *perturbation* of the wave train; a sudden beginning, a sudden ending, a transient swelling of the sound. Most measurements of the speed of sound, in fact, are measures of the speeds of something violent—the crack of a pistol or an electric spark, the roar of an explosion—something very unlike a uniform train of sine-waves.[5]

Now a sine-wave with a perturbation is in effect a sum of two or more sine-waves each of endless extent and constant amplitude, but having different wave-lengths and different amplitudes. This statement is the content of Fourier's principle from which the method of Fourier analysis is derived. One might represent even the sudden and violent pulsation of air due to an explosion, or the electrical spasm due to an outburst of static, by a summation of properly-chosen endless monochromatic sine-wave trains. I take however the simplest conceivable case: the wave train composed of only two sine-waves of different wave-lengths.

The reader will probably recall that when the difference between the wave-lengths is only a small fraction of either, this composite wave train resembles a sine-wave with regular fluctuations of amplitude—that is to say, with "beats" (Fig. 3). The maximum or centre of a beat occurs where a crest of one sine-wave coincides with a crest of the other—the minimum between beats, where crest falls together with trough. Denote the two wave-lengths by λ and $\lambda + \Delta\lambda$. One sees by inspection that a wave-length is the same fraction of the distance D between two consecutive beat-maxima, as the discrepancy $\Delta\lambda$ is of the wave-length:[6]

$$D/\lambda = \lambda/\Delta\lambda. \tag{17}$$

Of course this statement is exactly true only in the limit of vanishingly small $\Delta\lambda$. We shall always stay close to this limit, though some of the following statements would be valid even otherwise.

[5] I except so-called measurements of the velocity of sound which are really measures of frequency and wave-length in stationary wave-patterns, these being then multiplied together.

[6] The principle of the vernier.

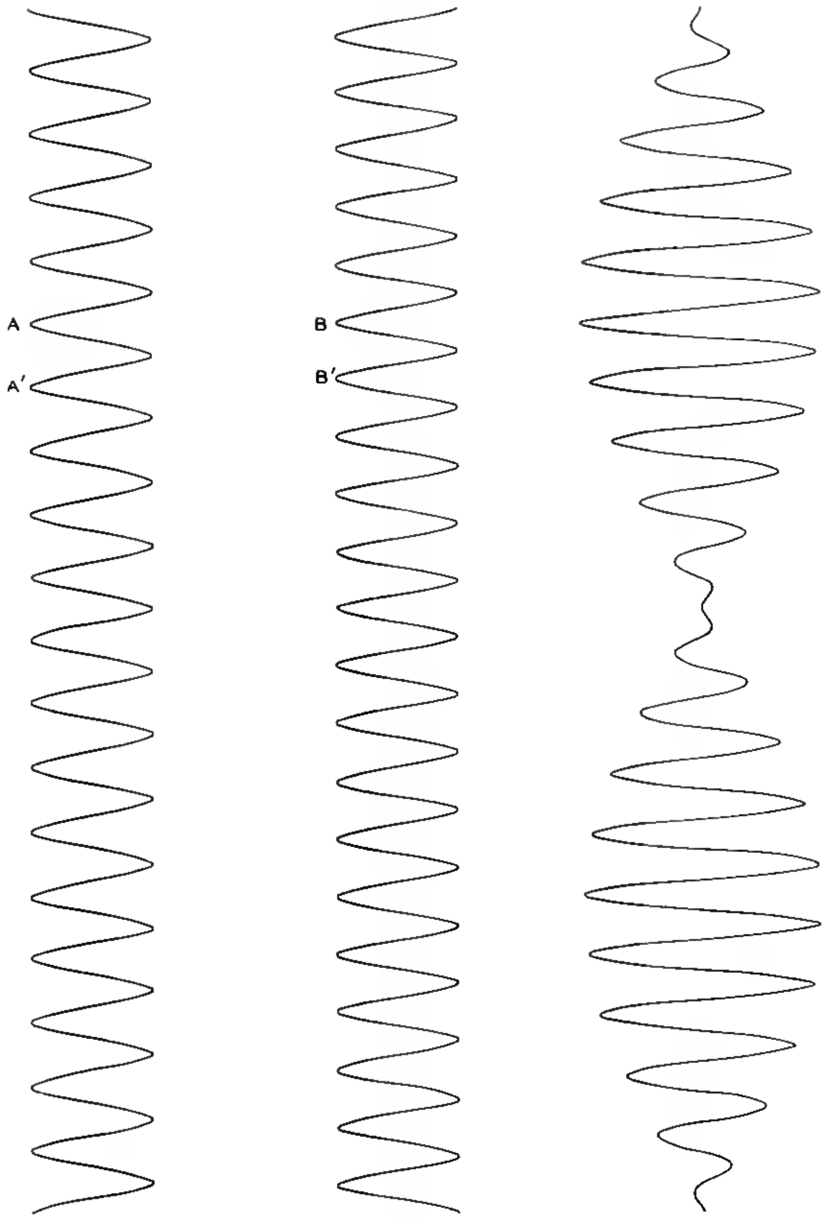

Fig. 3.

Now if the two component waves advance with equal speed, the beats are simply carried along with a speed equal to theirs. But if the velocities of the two component waves are not the same, then the velocity of the beats is not the same as either, nor the mean thereof. It is in fact something totally different.

To see this, imagine that you are moving along with one of the sine-waves; for definiteness, that you are riding on the crest B of the train with the shorter waves (Fig. 3). At a certain moment, say $t = 0$, it coincides with a crest A of the other sine-wave, and you are at the top of the beat. Meanwhile the other train is moving relatively to the first; for definiteness suppose that the longer waves move faster, so that relatively to the shorter they are gliding upward. After a certain time they have gained on the shorter waves by a distance $\Delta\lambda$, the difference between the two wave-lengths. But when this time has elapsed, the top of the beat is no longer where you are, but where the crest B' of the first train coincides with the crest A' of the second. It has dropped back through the distance λ, while the second wave train was getting ahead by the distance $\Delta\lambda$. Perhaps it will be easier to realize that while the second wave train is gaining on the first by λ, the beat is dropping back by the distance D between consecutive beats; by equation (17) this comes to the same thing.

Therefore when the longer waves travel faster than the shorter, the beats travel more slowly than either. If the longer waves were the slower, the beats would travel more rapidly; but this case is never realized in nature, not at least with light-waves [7] and waves of electricity and matter.

We now deduce the formula for the actual value of the speed of the beats. Denote by v and $v + \Delta v$ the speeds of the two sine-waves of which the wave-lengths are λ and $\Delta + \Delta\lambda$, respectively; by g the speed of the beats. It is sufficient to put into notation what has just been said in words. Relatively to the former wave train, the velocity of the latter wave train is Δv, that of the beats is $(g - v)$. Relatively to the former wave train, the latter moves a distance $\Delta\lambda$ while the beats are moving a distance λ in the opposite sense, therefore with a minus sign. Hence:

$$(g - v)/\Delta v = -\lambda/\Delta\lambda \tag{18}$$

[7] The exception to this statement—the case of light having wave-lengths lying within a region of anomalous dispersion of the transmitting substance—has been analyzed by Sommerfeld and L. Brillouin (*Ann. d. Phys.* **44**, pp. 177-202, 203–240; 1914) who find that in this case the group-speed defined by (20) loses its physical importance, and a segment of a wave train is transmitted with a speed never exceeding the speed of light in vacuum. This appears to be related to the absorption which always goes with anomalous dispersion.

and solving for g,

$$g = v - \lambda \frac{\Delta v}{\Delta \lambda}, \tag{19}$$

or going over to the differential notation, which will not only look more natural but will signify that the result which we have just attained is strictly valid in the limit for infinitesimal differences of wave-length:

$$g = v - \lambda(dv/d\lambda). \tag{20}$$

This is the formula for the *group-speed;* for the term "group-speed" is the usual one for what I have been calling "speed of beats." Likewise *phase-speed* is commonly used to denote the speed of the individual sine-wave trains.

The term "group-speed" is in one respect unfortunate; for it implies that any "group," that is to say any sequence of uneven and irregular wave-crests and troughs, is propagated with a perfectly definite speed. However this is true only for the simplified group which we have been considering, the beat formed of no more than two wave trains; and even for this it is exactly true only in the limit, where the wave-length-difference between the trains approaches zero. All other groups change in form as they advance. Now there is always something arbitrary in defining "speed" for something which changes as it goes, like a puff of smoke or a cloud. The arbitrariness is nil in only the limiting case which I have just been formulating. However, it must not be exaggerated. A bunch of irregular crests and troughs may retain enough of its form and compactness, as it travels over a distance many times as great as its width, to justify the statement that it has a speed of its own. And if such a group turns out, on being analyzed in Fourier's way, to consist mainly of sine-waves clustered in a small range of wave-lengths, then its speed will not be far from the value of g computed by equation (20) for a wave-length in that range.

Now these deductions explain a very remarkable experiment by Michelson, which otherwise might have disproved—indeed I do not see how it could have been interpreted otherwise than as destroying—*both* the wave and the corpuscle theory of light. I will preface the account of this experiment by saying that for light in empty space the speed of all wave-lengths is the same,[8] so that there never is any dif-

[8] The chief evidence for this statement is astronomical. If light of one color traveled faster than light of another, a luminous star emerging from behind a dark one would be seen first in the faster-travelling hue; in fact there would be a sequence of colors, the same for every emergence of every such star, and spread out over a time-interval proportional to the distance of the stars. Nothing of the sort has ever been observed, although there are plenty of luminous stars revolving around dark ones which regularly occult them.

ference between velocity of groups and velocity of wave-crests; they both have the same universal constant value c. However this cannot be true for light in transparent material media such as glass, water, or carbon bisulphide; for the refractive index of all these media varies from one wave-length to another—they are said to be *dispersive*.

Now Michelson measured the time taken by a flash of light to cover a measured distance, first through air (very nearly the same as vacuum) then partly through air and partly through carbon bisulphide. The source of light shines continuously, and an incessant beam falls on a revolving mirror and is reflected in a continuously-changing direction; a second, stationary mirror receives this reflected beam during a very small fraction of each complete revolution and sends it back, so that the twice-reflected beam is a series of segments cut from the primary beam. It was the time taken by the segments to travel a known distance which Michelson measured.[9] Reasoning back from the data, he computed that they took (1.76 ± 0.02) times as long to go a given distance in carbon bisulphide as in air. But the refractive index of carbon bisulphide, in the range of the spectrum where Michelson's source of light was brightest, is about 1.63; so that the primitive wave-front-theory gives 1.63 for the ratio of the speeds in air and CS_2, and the corpuscle-theory gives $(1.63)^{-1}$.

Foucault and Fizeau, be it remembered, had done the experiment with water. It happens that for water the derivative $dv/d\lambda$ is much smaller, and the group-speed therefore much closer to the wave-speed, than for carbon bisulphide. Also their experiments, though performed by the same method as Michelson was later to adopt and adapt, were less accurate than his. But if they had performed the Michelson experiment in 1850, the result would have been astounding. For Arago had asked, in effect: is it the speed of the wave-fronts in the wave-theory, or the speed of the corpuscles in the corpuscular theory, which agrees with the measured speed of a piece of light? Arago had said: "These experiments . . . will permit no further hesitation as between the rival theories. They will settle *mathematically* (I employ this word on purpose) they will settle mathematically one of the greatest and most disputed questions of natural philosophy." He had proposed a question to Nature, and had written down two and only two answers. Everyone thought that Nature must reply by ratifying one of the

[9] When the segments returned from the second to the first mirror they found that the latter had revolved a little further beyond the orientation which it had when they left it, so that it reflected them onward not quite along the path to the source of light, but along another path inclined to that one at an angle twice as great as that through which it had revolved. Michelson measured the angle, and knowing the rate of revolution of the revolving mirror he then knew how long the light had taken to go from it to the stationary mirror and back.

answers. Foucault and Fizeau reported that she had replied: *the former*. But they had not heard distinctly; for her actual response was: *neither*.

Michelson's experiment however came after the idea of group-velocity as distinguished from wave-velocity had been invented and established. The refractive index of carbon bisulphide varies with wave-length. On determining the wave-speed or phase-speed v from the refractive index (by the equation $N = c/v$) and then the derivative $dv/d\lambda$, it is found[10] that in the region of the visible spectrum, the term $\lambda(dv/d\lambda)$ amounts to about seven per cent of the term v, on the right-hand side of equation (20)—that is, the group-speed should be some seven per cent lower than the wave-speed in carbon bisulphide. In air, however, group-speed and phase-speed are sensibly the same. The ratio of the group-speeds in air and CS_2 falls close to Michelson's value.[11]

Coming as it did, therefore, the Michelson experiment merely showed that those who had subtilized the Huyghens' theory by introducing sine-waves had incidentally invented something able to move with the measured speed of a light-flash, though nothing of the sort had been available in the original form. Had it come earlier—well, there is no way of knowing what would have been inferred; but people might have come to think that after all a wavefront-theory or a corpuscle-theory of light may have some use and value, even though the speeds assigned to the waves or the corpuscles do not agree with those actually measured. Such an attitude of mind would be rather advantageous, today. As a corollary for the present I submit: in picturing a jet of free negative electricity as a beam of waves or a stream of corpuscles, we should not be too confident that either the speed of the waves or the speed of the corpuscles is the speed with which a segment dissected from the jet would move from place to place, until someone succeeds in making actual measurement of this last. Fundamental theory has something to say on this point, which we will presently consider.

[10] I take all the numerical values in this section from a review of Michelson's work by J. Willard Gibbs (*Am. Jour. Sci.* **31**, pp. 62–64; 1886) which so far as I know is the latest critical discussion of the data.

[11] The problem is more complex than I have intimated, not only because Michelson observed light covering a very wide range of wave-lengths so that v and $dv/d\lambda$ both extend over wide ranges of values, but also because different parts of a wave-front are reflected from different parts of the mirror at different moments, and therefore from *differently-inclined* parts. Quite a controversy went on during the eighteen-eighties in the pages of "Nature" as to what it was that Foucault had really measured. Rayleigh at first (*Nature* **24**, p. 382; 1882) thought it was g; then changed his mind, (**25**, p. 52; 1882) and decided it was v^2/g; then was convinced by Schuster (**33**, pp. 439–440; 1886) that it was really $v^2/2(v - g)$. J. W. Gibbs then took a hand (**33**, p. 582; 1886) and contended that after all it was really g. The controversy seems to have rested there. It may be added that Michelson's data eliminate v^2/g, but do not quite discriminate between g and Schuster's expression.

Group-Speed and Corpuscle-Speed

Thus far I have said that if we wish to use wave-theory and corpuscle-theory alternatively, we must make the momentum of the corpuscle equal to the quotient of the constant h by the wave-length of the waves; but I have said nothing about the energy of the corpuscle.

Let us adopt the universal assumption—based on a multitude of experiments, for instance those on the photoelectric effect—that the energy E of a corpuscle of light is equal to the product of its frequency ν by the same universal constant h; and let us extend it to the other kinds of corpuscles which we may associate with other kinds of waves, and *vice versa*.

Then the complete description of the particles associated with waves of wave-length λ is as follows:

$$p = h/\lambda, \qquad E = h\nu = hv/\lambda. \tag{21}$$

Here, as before, v stands for the phase-speed of the waves (not the particles).

Returning to the formula (20) for the group-speed, we now can write it thus:

$$\begin{aligned} g &= v - \lambda(dv/d\lambda) = \nu\lambda - \lambda d(\nu\lambda)/d\lambda \\ &= -\lambda^2(d\nu/d\lambda) = -(\lambda^2/h)(dE/d\lambda). \end{aligned} \tag{22}$$

Suppose next that the energy and the momentum of the corpuscles in question are related to each other and to their speed in the well-known fashion of ponderable bodies, to which it is known that electrons conform. Thus for sufficiently low speeds, the relations are practically those of the "classical" mechanics:

$$p = m_0 u, \qquad E = \tfrac{1}{2} m_0 u^2, \text{ whence } E = p^2/2m_0. \tag{23}$$

Here m_0 stands for the constant mass, u for the speed of the corpuscles (*not* the waves).

The energy of the corpuscles is a function of the momentum only, and continuing to develop the formula (22) for the group-speed, we find:

$$\begin{aligned} g &= (-\lambda^2/h)(dE/dp)(dp/d\lambda) = dE/dp \\ &= p\sqrt{1-\beta^2}/m_0 = u. \end{aligned} \tag{24}$$

The group-speed of the waves is equal to the speed of the corpuscles.

The same conclusion follows if we use the relativistic definitions for the energy and the momentum of a particle,

$$E = m_0c^2/\sqrt{1-\beta^2}, \quad p = m_0\beta c/\sqrt{1-\beta^2} \qquad (\beta = u/c),$$
$$E = c\sqrt{m_0^2c^2 + p^2}$$

as the reader may test for himself.

Summarizing: if the corpuscles associated with the waves have the properties of ordinary material bodies—*if*, let us say, for short, *the corpuscles are material particles, their speed is equal to the group-speed of the waves.*

This is a very happy and agreeable result. It compensates very largely for our having been forced to concede that if we want both waves and corpuscles, the wave-speed and the corpuscle-speed must be different. The wave-theory has supplied another velocity which is equal to that of the corpuscles. Moreover it is precisely the velocity with which we should expect an isolated segment of a wave train to move from place to place. If someone were to cut a piece out of an electron-jet and measure the time it took to traverse a known distance, the speed which he would deduce from his data would probably agree both with the corpuscle-speed and with the group-speed, and disagree with the wave-speed. It would be interesting to try this out.

In the equations (23) I have taken account only of the kinetic energy of the corpuscles; in the equations (25), only of their kinetic energy and of the "rest" energy associated with their mass. But the explanations of refraction by the two theories will no longer be concordant, unless the potential energy also is admitted. Let us denote the potential energy of a corpuscle by U; and, since as yet these theories have been verified only for negative electricity, let us immediately write eV for U, e standing for the charge of an electron and V for the electrostatic potential in the region where it is. For the total energy of the corpuscle, then, we have instead of (25) the relativistic expression,

$$E = m_0c^2/\sqrt{1 - \beta^2} + U = m_0c^2/\sqrt{1 - \beta^2} + eV, \tag{26}$$

which for small values of the corpuscle-speed u ($= \beta c$) reduces to the classical expression,

$$E = \tfrac{1}{2}m_0u^2 + U = \tfrac{1}{2}m_0u^2 + eV. \tag{27}$$

In an earlier section we interpreted the refraction of an electron beam passing from vacuum into metal by thinking of the metal and the vacuum as being two regions in which different values of electrostatic potential prevail, the potential thus changing sharply from one value to the other at the surface which bounds the solid. Now when the beam considered as a stream of corpuscular electrons passes across such a surface, the energy of each electron as expressed by (26) or (27) remains the same, though the proportion which is kinetic energy is changed; and therefore the frequency E/h of the equivalent wave-train remains the same. If then we keep the assumption that the wave-

length of the waves is equal to h divided by the momentum of the particles, we have the following value for the ratio between the wave-speeds v' and v on the two sides of the surface:

$$v/v' = \nu\lambda/\nu\lambda' = \left(\frac{E}{h}\frac{h}{p}\right)\Big/\left(\frac{E}{h}\frac{h}{p'}\right) = p'/p, \tag{28}$$

and the speed of the waves varies inversely as the momentum of the corpuscles, which is just what is required in order that we may hold both the theories simultaneously.

But how about the theorem that corpuscle-speed is equal to group-speed? Returning to the equations (25), we see that the introduction of the potential energy has altered the relation between energy and momentum; we now have:

$$E = c\sqrt{m_0c^2 + p^2} + eV. \tag{29}$$

But so long as we are comparing different electron-streams in the same medium (vacuum, for instance), the potential energy is the same for all and does not depend on the momentum; and differentiating E with respect to p to obtain the value of the group-speed g, we get:

$$g = dE/dp = \frac{c^2p}{E - eV} = \frac{c^2m_0u/\sqrt{1-\beta^2}}{m_0c^2/\sqrt{1-\beta^2}} = u, \tag{29}$$

and thus group-speed and corpuscle-speed are equal, as before.

I will write down the expression of the phase-speed, although for the physicist it is of minor importance, not being measurable—a fact which exempts us, temporarily at least, from pondering over the curious feature that it depends on the value of the potential energy of the corpuscles, and therefore (for electrons) on the value accepted for the electrostatic potential of the region where the wave-train is, even though in practice it is generally assumed that electrostatic potential may be measured from an *arbitrary* zero. The formula is this:

$$\begin{aligned} v = E/p &= \frac{m_0c^2/\sqrt{1-\beta^2} + U}{m_0u/\sqrt{1-\beta^2}} \\ &= c^2/u + U/p, \end{aligned} \tag{30}$$

and if we put the potential energy of the corpuscles equal to zero, we find the phase-speed varying inversely as the corpuscle-speed,[12] and greater than the speed of light.

[12] There is a paradox here which, as I can testify from personal experience, is a dangerous source of confusion. The formula $v = c^2/u$ sounds like an approximation to the formula $v = const/p$ which I have given as the requisite relation between

Stationary Waves and Oscillating Particles

We have tried out, separately and in tandem, two alternative ways of interpreting a beam of radiation advancing through space; first as a stream of corpuscles, then as a train of waves. We will now try out two alternative ways of interpreting radiation enclosed in a box; first as a system of stationary waves, then as a quantity of corpuscles rushing to and fro and bouncing from the walls. To simplify the case as much as possible, think only of motions parallel to one side of the box; or to make the pictures more graphic, think of a tube or pipe like those often used in experiments on sound, in which the waves travel along the axis.

Now it is well known that when a train of sound-waves is sent through a tube, or generated by vibrations somewhere in the tube, it is partially reflected from the far end, then again partially reflected from the near end, and so on over and over again; the overlapping wave trains passing to and fro interfere with one another; and when the wave-length is related in a certain way to the length of the tube, the overlapping wave trains form a *stationary wave-pattern* of alternating loops and nodes—the tube is said to be in resonance. If the two ends of the tube are alike (both open, or both closed) so that reflection takes place in the same way as both, the waves which admit of resonance are those of which the half-wave-length or an integer number of half-wave-lengths fits exactly into the tube; denoting by d the length of the tube, these wave-lengths are given by the formula:

$$n\left(\frac{\lambda}{2}\right) = d, \qquad n = 1, 2, 3, \ldots \tag{41}$$

This equation defines what may be called the *characteristic wave-lengths* of the tube. The tube distinguishes these, or the wave trains possessing these wave-lengths, from all the others.

Suppose on the other hand we had particles rushing back and forth along the axis of the tube, and rebounding without loss of energy whenever they struck either wall. Denote by u the speed of a particle; it takes a time-interval $2d/u$ to describe a complete round-trip with two rebounds, and one might say crudely that it has a frequency $u/2d$. I say "crudely" because the corpuscle is not moving with a sinusoidal motion, like a pendulum-bob; its speed does not vary as a sine-function

wave-speed and momentum. However the two relate to entirely different situations. The first is a comparison between wave-speeds and corpuscle-speeds for different beams in the same medium. The second is a comparison between wave-speeds and corpuscle-momenta for the same beam in different media. The resemblance between the two is accidental and misleading.

I am indebted to Professors C. H. Eckart and E. C. Kemble for elucidation of this point.

of time, but retains the same value throughout except for the change of direction; if we were to apply a Fourier analysis to this motion, we should find not only the frequency $u/2d$, but all of its overtones. Let us think however only of this fundamental frequency. Now it is evident that there is nothing, in our ordinary conceptions of particles rushing back and forth and rebounding from walls, to distinguish any value of speed or frequency above any others. The phenomenon of resonance sets certain wave-lengths apart from others, but there is nothing to correspond to resonance in this latter case, and set certain speeds apart from others.

But instead of sound, think of some kind of radiation which we have interpreting both as corpuscles and as waves—light, for example. Light enclosed between parallel reflecting walls forms stationary waves,[13] provided that its wave-length is related to the distance d between the walls by the equation (41), which I rewrite:

$$\lambda = 2d/n, \qquad n = 1, 2, 3 \ldots \tag{42}$$

The parallel reflecting walls, or the limitation which they set upon the space accessible to the light, thus single out certain characteristic wave-lengths and distinguish them from all others. How interpret this fact by corpuscle-theory?

Well, we have been associating waves of wave-length λ with corpuscles of momentum $p = h/\lambda$; let us continue to do so. The reflecting walls, then, single out certain characteristic values of momentum given by this equation, derived straight from (42):

$$p = nh/2d, \tag{43}$$

which I proceed to rewrite thus,

$$2d \cdot p = nh \qquad n = 1, 2, 3 \ldots \tag{44}$$

These values of momentum, I have said, are set apart from all the rest. If waves and corpuscles are interchangeable as bases for a theory of light, then the feature of wave-motion known for short as "resonance" obliges us to make that supposition. But in what way, and to what extent, are they set apart? According to modern quantum-theory, they are actually the *only* possible values. A particle describing a cyclic motion of this character, in which it moves a fixed distance with a fixed momentum and then moves the same distance backward with the same momentum reversed and so forth ad infinitum, is constrained by something in the order of nature to have one or

[13] Interference patterns are essentially of this type, though usually they are formed between mirrors oblique to one another.

another of the "permitted" momenta defined by equations (43) and (44).

Examining equation (44), one sees how this definition of the permitted momenta may be stated. The quantity on the left of (44) is the product of the momentum of the particle, by the distance which it traverses each time it performs its cycle.[14] This product must be equal to an integer multiple of the Planck constant h.

Now the quantum-theory of the atom developed fifteen years ago by Bohr, Sommerfeld and W. Wilson—the first and greatest of the forward steps in the contemporary conquest of the problem of atomic structure—was based on the assumption that an electron performing a cyclic motion must perform it in such a way, that its momentum conforms to a condition of which equation (44) is but a special case. This is the condition always written thus:

$$\int pdq = nh, \qquad n = 1, 2, 3 \ldots \tag{45}$$

If the electron is oscillating to and fro in a straight line through a position of equilibrium, q stands for its distance from that position and p for its momentum, and the integral is taken once around a complete oscillation. It is evident that (44) is the special form of this equation for the case in which the force acting on the electron is vanishingly small until it hits the wall and then suddenly becomes enormous. If the electron is revolving in an orbit in two or three dimensions, there are two or three equations like (45) all postulated at once; but I shall not take up such more complicated cases.

Summarizing the outcome of this section in a phrase: *if we associate waves of wave-length λ with corpuscles of momentum h/λ, and stationary waves in an enclosure with corpuscles flying back and forth between its walls, then the condition that the waves must fulfil to form a stationary system is equivalent to the quantum-condition imposed upon the corpuscles.*

This is an illustration of wave-mechanics. How extraordinarily fruitful and valuable such comparisons have proved in the hands of Louis de Broglie, of Schroedinger, Bose, Fermi and Sommerfeld—to name only a few—I have shown in part, in earlier issues of this journal. Here it must suffice to say that Schroedinger developed the principle into a form suitable for predicting the stationary states of atoms; Bose constructed out of it a competent theory of radiation in thermal equili-

[14] It travels a distance d in the forward sense with a momentum p, and then an equal distance in the backward or negative sense with a momentum of equal amount but reversed sign, so that the total product of distance by momentum is

$$pd + (-p)(-d) = 2dp.$$

brium, considered as a gas of which the atoms are corpuscles of light; while Fermi, Dirac and Sommerfeld between them used it to make a powerful theory of the free negative electricity in metals, conceiving this alternatively as a gas of which the atoms are electrons, and a system of stationary waves enclosed within the surface of the metal as in a box with reflecting walls.

Diffraction of Waves and Diffraction of Corpuscles

The effect of a diffraction-grating upon a beam of light projected against it has always been considered the most striking evidence that light is of the nature of waves and not of corpuscles. Indeed it is considered to suffice in itself to prove the corpuscle-theory untenable. With any common understanding of the term *corpuscle-theory*, this statement is correct; but we had better put it in the softer form, that the effect of a diffraction-grating on a beam of light proves that if we adopt a corpuscular theory we must endow the corpuscles with some very strange property which nobody ever thought that particles could possess, and which may even seem to be in contradiction with their nature. We had better put the statement in this milder way, because it now is known that in spite of all the evidence for individual electrons, a beam of negative electricity is affected by a grating in much the same way as a beam of light.

Take then almost the simplest conceivable case of diffraction; a plane-parallel beam of light falling perpendicularly on a wall containing many equally-spaced parallel slits, and a part of the light passing through the slits to a screen infinitely far away. On this infinitely-distant screen—which may in practice be brought up to a convenient nearness, by means of a lens—one sees a peculiar pattern of light and shade. I single out one particular feature of this pattern: the fact that there are maxima of illumination along certain lines parallel to the slits. One of these, for instance, is straight ahead from the slits, along the direction of the incident beam prolonged; another is off to one side, in a direction making a certain angle (say ϕ) with that of the incident beam; another is equally far off to the other side. These two last-named, the *first-order maxima*, are those we shall consider; it will be enough to speak of one.

By the wave-theory, a first-order maximum is explained as follows. Each of the slits is the source of a secondary wave train of spherical wave crests, stimulated by the primary wave train, and having the same frequency and wave-length. Consider any two adjacent slits. Secondary wave crests start from the two at the same moment. At any point equally distant from the slits, they arrive simultaneously, and

reinforce each other; this is the explanation of the central bright fringe. At any point not quite equally distant from the slits, they do not arrive quite simultaneously, and the reinforcement is impaired. But at a point which is further from one slit than from the other by just the wave-length λ, the wave crest arriving from the latter meets the next previous crest from the former, and the reinforcement is restored. The first-order maximum is located at these points.

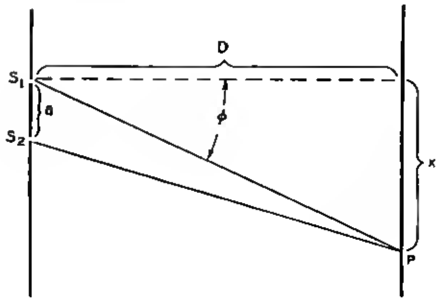

Fig. 4.

From Fig. 4 one sees[15] that when the screen is very far away, the points distant from the slit S_1 by one wave-length more than they are distant from S_2 are situated in the direction inclined at ϕ to the straight-ahead direction, the angle ϕ being given approximately by the formula

$$\sin \phi = \lambda/a, \tag{46}$$

where a stands for the distance between the slits. When the screen is infinitely far away, the formula is exact. (I must admit that it is somewhat disingenuous to simplify the problem by solving only the special case in which the screen is infinitely far away, for the general case opposes much more serious difficulties to the corpuscle-theory; but this is the special case of greatest physical importance, and one has to make a beginning somewhere.)

We have now explained the presence of a first-order maximum in the pattern of light and shade on the screen, though it cannot be said that we have "verified" formula (46), for that formula serves as the practical definition of wave-length; wave-lengths are measured by

[15] From the figure we see that for d_1 and d_2, the distances from S_1 and S_2 to the point P on the screen, we have:

$$d_1^2 = D^2 + x^2, \qquad d_2^2 = D^2 + (x - a)^2, \qquad d_1 = D \sec \phi, \qquad x = D \tan \phi$$

and hence

$$(d_1 - d_2)(d_1 + d_2) = 2ax - a^2.$$

When D, x, d_1 and d_2 all become infinite together, the second factor on the left becomes equal to $2D \sec \varphi$ and the second term on the right may be neglected.

measuring the angle ϕ and using equation (46). Let us now try the corpuscle-theory on the problem.

Putting as heretofore the value h/λ for the momentum of the corpuscles, translate (46) into the language of the alternative theory; one gets:

$$\sin \phi = h/ap. \tag{47}$$

In words: a corpuscle of momentum p, passing through any slit, is particularly likely to bend around through an angle ϕ of which the sine depends in a certain way on its momentum and on the distance to the next slit.

Which is to say: the likelihood that a corpuscle entering a slit will bend its course through a certain angle depends on the presence of other slits in the same wall, and on the distance between these slits.

But the reader will inquire: how does the corpuscle entering one of the slits know that the other slits are there? If all the other slits were suddenly stopped up, the first-order maximum would vanish, the likelihood that the corpuscle would turn in the direction given by (47) would fall to zero; but how could it know that they had been stopped up?

Well! this is precisely the strange and extravagant property with which we are forced to endow the corpuscles, if we want to use the particle-theory to explain diffraction. It must be supposed that when passing through a slit, a particle of light knows whether there are other slits and, if so, how they are spaced. It must be supposed that an X-ray particle striking an atom in a crystal knows that there are other atoms in a regular array, and knows moreover just the pattern and the scale of that array. It must be supposed that electrons enjoy a like omniscience. Or to express it in more technical language; the probability that a corpuscle of light, of electricity or of matter shall be deflected through a given angle when it strikes an atom or passes through a slit must be supposed to depend on the arrangement of the other atoms or the other slits in the vicinity. This idea is not easy to accept; but it must be accepted, if one is to build up a complete corpuscular theory of any of these entities.

But if one accepts it, one finds that the stipulation (47) turns out to be another example of the general quantum-condition of which, in (44), we have already met one instance. For write it thus:

$$ap \sin \phi = ap_t = nh, \qquad n = 1, 2, 3 \ldots, \tag{48}$$

the factor n being now introduced to take account of the maxima of second, third, and higher order which also occur on the screen, though

I refrained from mentioning them earlier. I have used the symbol p_t for the quantity $p \sin \phi$, for this, as one sees immediately, is the tangential component of momentum which the corpuscle acquires at the deflection, not having had any before. The wall containing the slits, or the row of atoms if we consider instead the diffraction of X-rays by a crystal, receives an equal momentum in the opposite sense. We may therefore say that diffraction occurs in such a way, that the regularly-spaced series of slits or atoms receives a momentum p_t given by the formula:

$$ap_t = nh. \tag{49}$$

But now what is the product ap_t? It is the product of the momentum of the row of atoms or slits by the distance a between any adjacent two; it is therefore the integral $\int pdq$ of the general principle (45), evaluated for the range of integration a. Now the general principle is supposed to apply when the range of integration covers a complete cycle of a periodic motion. There is nothing obviously periodic about a steady sidewise sliding of a row of atoms with a constant momentum. But in a sense, there is after all something periodic. For if the row of equally-spaced atoms (or slits) extends to infinity in both directions, then when it has moved sidewise through the distance a each atom lies exactly in the former place of another atom, and the original arrangement is to all appearances restored. The steady onward motion of the regular array is also a cyclic departure and return to a periodically-restored arrangement; and the maxima of the diffraction-pattern are determined by applying the quantum-condition to this cyclic motion.

The reader may ask: how about the component of momentum in the direction at right angles to the grating? Without precisely answering that question, I will end the article by applying the corpuscular theory to a case in which all the components of momentum are duly taken into account: diffraction of X-rays or of electrons by a three-dimensional crystal.

Suppose an "ideal" crystal extending infinitely far in all directions. It is composed of similar and similarly-oriented "atom-groups"—I will use the language and the symbols of the eighteenth article of this series—arranged upon a "space-lattice," of which the three spacings shall be denoted by a, a', a''. If we start with one atom-group A, then along one direction from it there is an infinite sequence of such groups at distances a. $2a$, $3a$, . . . and also at distances $(-a)$, $(-2a)$, $(-3a)$, . . . in the opposite sense. Call that the x-direction. Then along another direction through A, say the y-direction, there is an

infinite sequence of groups at distances a', $2a'$, $3a'$, . . . and $(-a')$, $(-2a')$, etc.; and along a third or z-direction through A, there is an infinite sequence of atom-groups spaced at intervals a''.

Now think of the atom-groups as hard particles, and the corpuscle of light or of electricity (the "X-ray quantum" or the electron) as a hard particle which rushes into the lattice, hits one of the atom-groups—A, say—and bounces off. Denote by ϕ, ϕ', ϕ'' the angles which its original direction of motion makes with the x, y, z directions respectively; by θ, θ', θ'' the angles which its final direction of motion makes with these three. Before the deflection, the corpuscle has a momentum of magnitude p, parallel to its original direction of flight; afterward it has a momentum of the same magnitude, but parallel to its final direction of flight. At the deflection, then, it loses—that is, it communicates to the lattice—a momentum of which the three components along x, y, z have the values:

$$p(\cos\theta - \cos\phi); \qquad p(\cos\theta' - \cos\phi'); \qquad p(\cos\theta'' - \cos\phi'').$$

Now if, following the foregoing procedure, we equate the first of these to some integer multiple of h/a, the second to some integer multiple of h/a', and the third to some integer multiple of h/a'', and then translate momentum of corpuscles into wave-length of waves by the now-familiar formula $p = h/\lambda$, we get:

$$\begin{aligned} a(\cos\theta - \cos\phi) &= n\lambda, \\ a'(\cos\theta' - \cos\phi') &= n'\lambda, \\ a''(\cos\theta'' - \cos\phi'') &= n''\lambda, \end{aligned} \tag{50}$$

where n, n', n'' stand for any three integers. Now these are the equations (numbered 3, 4, 5 in the eighteenth article) to which conform the "Laue beams," which is to say, the directions in which electrons and light are actually diffracted by crystals.

Perhaps I should close with two or three admonitions. To make the wave-theory and the corpuscle-theory equivalent for a few simple cases is of course not at all the same as making them equivalent universally. Also, the examples in this article are not always so elementary as they may seem. The first involved two distinct media with a sharp boundary between; and discontinuity is always less agreeable than continuity to the mathematician. The last but one involved a non-sinusoidal vibration, which is much more complex than a sinusoidal one. Moreover, the concepts of light-waves and quanta are not nearly so beautifully welded together as those of electricity-waves and electrons. Nevertheless these illustrations may help to weaken the idea that there is no way out of the present situation but to abandon either waves or corpuscles; for decidedly, there is a way.